Finite Mathieu Transform on Elastic Vibration in Elliptical Plate

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Abstract-

This paper is concerned with elastic vibration in an elliptical plate to determine the vibration of plate at $\xi = \xi^{(1)}$ *for time (t > 0) with the help of finite mathieu transform technique.*

Keywords – Elliptical plate, Mathieu transforms, Marchi-Fasulo transform, Laplace transform, Elastic vibration.

Introduction-

In this paper, an attempt has been made to determine the *elastic vibration in a elliptical plate* with known radiation type boundary conditions, using Mathieu transform, Laplace transform and Marchi-Fasulo transform technique.

Statement of The Problem

Heat conduction equation in elliptical co-ordinates (ξ,η) for elliptical cylinder define as

$$b^2\left(\frac{\partial^2\omega}{\partial\xi^2}+\frac{\partial^2\omega}{\partial\eta^2}\right)\left(\frac{2d^2}{\cosh 2\xi-\cos 2\eta}\right)+\frac{\partial^2\omega}{\partial t^2}=\frac{P(\xi,\eta,t)}{2Pd} \tag{1}$$

Subject to the interior condition

$$\omega=(\xi,\eta,t)=f(\eta,t), \quad \text{(Unknown)}$$
$$0\le\xi\le a, \quad -h\le t\le h \tag{2}$$

The boundary conditions are

$$\left[\omega(\xi,\eta,t)+K_1\frac{\partial\omega(\xi,\eta,t)}{\partial t}\right]_{t=h}=F_1(\xi,\eta) \tag{3}$$

$$\left[\omega(\xi,\eta,t)+K_2\frac{\partial\omega(\xi,\eta,t)}{\partial t}\right]_{t=-h}=F_2(\xi,\eta) \tag{4}$$

$$\left[\omega(\xi,\eta,t)\right]_{\xi=a}=g(\eta,t) \quad \text{(Unknown)} \tag{5}$$

where k_1 and k_2 are radiation constants on the plane surfaces of the cylinder. Equations (1) to (5) constitute the mathematical formulation of the problem under consideration.

Solution of The Problem

We define the transform with respect to t as

$$\overline{\omega}(\xi,\eta,\varsigma)=\int_{-h}^{h}\omega(\xi,\eta,t)P_m(t)dt \tag{6}$$

Its inverse transform is

$$\omega(\xi,\eta,t)=\sum\frac{\overline{\omega}(\xi,\eta,\varsigma)}{\lambda_m}P_m(t) \tag{7}$$

Applying the transform defined in (6) to the equations (1),(2), (5) and using (3),(4) one obtains

$$\left(\frac{2\bar{d}^2}{\cos(2\xi)-\cos(2\eta)}\right)\left(\frac{\partial^2\overline{\omega}}{\partial\xi^2}+\frac{\partial^2\overline{\omega}}{\partial\eta^2}\right)=a_m{}^2\overline{\omega}+P.I. \tag{8}$$

Where the eigen values a_m are the solutions of the equation

$$\left[\alpha_1 a\cos(ab)+\beta_1\sin(ab)\right]\times\left[\beta_2 a\cos(ab)+\alpha_2 a\sin(ab)\right]$$
$$=\left[\alpha_2 a\cos(ab)-\beta_2\sin(ab)\right]\times\left[\beta_1\cos(ab)-\alpha_1 a\sin(ab)\right]$$

$$\overline{\omega}(\xi,\eta,\varsigma)=\bar{f}(\eta,\varsigma) \tag{9}$$

Where $\overline{\omega}$ denotes the Marchi-Fasulo transform of ω and ς denotes the Marchi-Fasulo transform parameter.

From equation (8), we obtain

$$\left(\frac{\partial^2\overline{\omega}}{\partial\xi^2}+\frac{\partial^2\overline{\omega}}{\partial\eta^2}\right)=2q\left(\cos(2\xi)-\cos(2\eta)\right) \tag{10}$$

where

$$q=\frac{a_m{}^2}{4d^2} \tag{11}$$

If temperature is symmetric about both axes of the ellipse, the appropriate solution of equation (10) is

$$\overline{\theta}(\xi,\eta,\varsigma)=\sum C_{2n}Ce_{2n}(\xi,-q)ce_{2n}(\eta,-q)\ +P.I \tag{12}$$

Where $Ce_{2n}(\xi,-q)$ and $ce_{2n}(\eta,-q)$ are defined as modified and ordinary Mathieu function of order 2n.

Using equations (9) and (12), we get

$$\overline{f}(\eta,\xi)=\sum C_{2n}Ce_{2n}(\xi,-q)ce_{2n}(\eta,-q) \tag{13}$$

In order to get the value of constant C_{2n} , multiply (13) by $Ce_{2n}(\eta,-q)$, integrate with respect to η from 0 to 2π and making use of the following result:

$$\int_0^{2\pi} Ce_{2n}^{\ 2}(\eta,-q)d\eta = \pi \quad \text{(Orthogonal properties),}$$

$$\int_0^{2\pi} C_{2n}(\eta,-q)\bar{f}(\eta,\varsigma)d\eta = \bar{f}_{2n}(\varsigma) \quad \text{(Say)} \tag{14}$$

We get,

$$C_{2n} = \sum_{n=0}^{\infty} \frac{\bar{f}_{2n}(\xi)}{\pi Ce_{2n}(a,-q)}. \tag{15}$$

Substituting equation (15) in equation (12), one obtains,

$$\bar{\theta}(\xi,\eta,\varsigma) = \sum_{n=0}^{\infty} \frac{\bar{f}_{2n}(\varsigma)Ce_{2n}(\xi,-q)ce_{2n}(\eta,-q)}{ce_{2n}(a,-q)} \tag{16}$$

Applying inverse transform defined in (7) to the equation (16) and using condition (5), one obtain the temperature distribution and unknown temperature gradient as,

$$\theta(\xi,\eta,t) = \sum_{m=0}^{\infty} \frac{P_m(t)}{\lambda_m} \sum_{n=0}^{\infty} \frac{\bar{f}_{2n}(\varsigma)Ce_{2n}(\xi,-q)ce_{2n}(\eta,-q)}{ce_{2n}(\xi,-q)} + P.I \tag{17}$$

$$g(\eta,t) = \sum_{m=0}^{\infty} \frac{P_m(t)}{\lambda_m} \sum_{n=0}^{\infty} \frac{\bar{f}_{2n}(\varsigma)Ce_{2n}(a,-q)ce_{2n}(\eta,-q)}{ce_{2n}(a,-q)} + \phi \tag{18}$$

Where,

$$\phi = [P.I]_{\xi=a}$$

$$\bar{f}(\xi,\eta,\varsigma) = \int_{-h}^{h} f(\xi,\eta,t)P_m(t)dt$$

and $\lambda_m = \int_{-h}^{h} P_m^{\ 2}(t)dt$

$$P_m(t) = Q_m \cos(a_m t) - W_m \sin(a_m t),$$

$$Q_m = a_m(\alpha_1 + \alpha_2)\cos(a_m h) + (\beta_1 - \beta_2)\sin(a_m h)\ ,$$

$$W_m = (\beta_1 + \beta_2)\cos(a_m h) + (\alpha_2 - \alpha_1)a_m \sin(a_m h)$$

The Mathieu transformation of $[f(\xi,\eta)]$

At $M[f(\xi,\eta)] = \overline{f}(q_{2n,m}) = \int_0^2 \int_{\xi_0}^{\xi^{(1)}} (\cosh 2\xi - \cos 2\eta)\ B_{2n}(\xi, q_{2n,m}) x\ ce_{2n}(\eta, q_{2n,m}) d\xi, d\eta$ (19)

Where

$$B_{2n}(\xi, q_{2n,m}) = [\{FeY_{2n}(\xi, q_{2n,m}) - FeY_{2n}(\xi^{(1)}, q_{2n,m})\}\times$$
$$xCe_{2n}(\xi, q_{2n,m}) - \{Ce_{2n}(\xi_0, q_{2n,m}) - Ce_{2n}(\xi^{(1)}, q_{2n,m})\}\times \quad (20)$$
$$x\ FeY_{2n}(\xi_0, q_{2n,m})]$$

$q_{2n,m}$ is the root of the equation

$$Ce_{2n}(\xi^{(1)}, q) FeY_{2n}(\xi_0, q) - FeY_{2n}(\xi^{(1)}, q) Ce_{2n}(\xi_0, q) = 0$$

And

$$FeY_{2n}(\xi, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \frac{Ce_{2n}(\xi, q)}{A_0^{2n}} Y_{2r}(2k' Sinh\xi) \quad , |Sinh\xi| > 0,\ R(\xi) > 0$$

Where k = q and Y is Bessel function

Thus integral transform possess the property

$$\omega(\xi_0, \eta, t) = M\left[\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) \frac{2d^{-2}}{(Cosh 2\xi - Cos 2\eta)}\right]^2 \omega(\xi, \eta) = \frac{-4q_{2n,m}}{d^2} \overline{\omega} \quad (21)$$

Inversion of Mathieu transform is

$$\omega(\xi, \eta,) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\overline{\omega} B_{2n}(q_{2n,m})}{\pi \int_{\xi_0}^{\xi^{(1)}} Ce_{2n}^2(\xi, q_{2n,m}) [\cosh 2\xi - \theta_{2n,m}] d\xi} \quad (22)$$

Where

$$\theta_{2n,m} = \frac{1}{\pi} \int_0^{2\pi} ce_{2n}^2(\eta, q_{2n,m}) \cos 2\eta\ d\eta \quad (23)$$

Now by applying the finite Mathieu transform (19) and using the equation (17) ,(21),(22) & (23) we get,

$$\frac{d^2\overline{\varpi}}{dt2}+b^2(\lambda^2{}_{2n,m})^2=\frac{\overline{P}(q_{2n,m,}t)}{2\rho d} \tag{24}$$

Where

$$\lambda^2_{2n,m}=\left(\frac{4q_{2n,m}}{d^2}\right) \tag{25}$$

Now this equation is solved by Laplace transform and the following result is obtained

$$\overline{\varpi}=\overline{f}(\lambda_{2n,m})\cos b\lambda^2_{2n,m}\ t+\frac{\overline{g}(\lambda^2{}_{2n,m})}{b\lambda^2{}_{2n,m}}\sin b\lambda^2{}_{2n,m}.t$$
$$+\frac{1}{2\rho db\lambda^2_{2n,m}}\int_0^1\sin b\lambda^2{}_{2n,m}.(t-\tau)x\overline{P}\ (\lambda^2{}_{2n,m},\tau)d\tau \tag{26}$$

Using inversions of Mathieu Transform we get

$$\omega(\xi,\eta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{2n,m}B_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m},t) \tag{27}$$

$$C_{2n,m}=\frac{\overline{\varpi}(q_{2n,m},t)}{\pi\int_{\xi_0}^{\xi^{(1)}}B^2_{2n}(\xi,q_{2n,m})[\cosh 2\xi-\theta_{2n,m}]d\xi} \tag{28}$$

Hence from equation (27) and (28) we get,

$$\omega(\xi,\eta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\overline{\varpi}(q_{2n,m},t)B_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m,}t)}{\pi\int_{\xi_0}^{\xi^{(1)}}B^2_{2n}(\xi,q_{2n,m})[\cosh 2\xi-\theta_{2n,m}]d\xi} \tag{29}$$

Where,

$$\overline{\varpi}=\overline{f}(\lambda_{2n,m})\cos b\lambda^2_{2n,m}\ t+\frac{\overline{g}(\lambda^2{}_{2n,m})}{b\lambda^2{}_{2n,m}}\sin b\lambda^2{}_{2n,m}.t+\frac{1}{2\rho db\lambda^2_{2n,m}}\int_0^1\sin b\lambda^2{}_{2n,m}.(t-\tau).\overline{P}\ (\lambda^2{}_{2n,m},\tau)d\tau$$

Equation (29) is the desired solutions of the problem

Conclusion

In this paper, we have investigated the elastic vibration of elliptical plate on the outer curved surface with the help of the finite Mathieu transform techniques. The results are obtained in the form of infinite series. The expressions that are obtained can be applied to the design of useful structures or machines in engineering application.

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